EFFECT OF FINITE VELOCITY OF THERMAL WAVE ON STRESS FOCUSING PHENOMENA

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1. Introduction

The analysis of a long cylindrical rod subjected to a sudden rise in temperature uniformly over its cross section has been studied by Ho [1]. Due to the instantaneous heating, the stress waves reflected from the cylindrical surface of the rod may accumulate at the center and give rise to very high stresses, even though the initial thermal stress is relatively small. This phenomenon is called the stress focusing effect. Hata has studied this effect for several cases of cylindrical rod [2-4]. The stress focusing effect for spheres has been studied by Mann-Nachbar [5] and Hata [6, 7]. However, these series of papers have been studied in the context of classical uncoupled theory of dynamic thermoelasticity.

The theory of dynamic thermoelasticity which takes into account the coupling effects between temperature and strain fields involves the infinite thermal wave speed. The theory of generalized thermoelasticity has been developed in an attempt to eliminate the physical paradox of the infinite velocity of thermal propagation. At present, there are two theories of the generalized thermoelasticity: the first is proposed by Lord and Shulman [8], the second is proposed by Green and Lindsay [9]. Recently, other theories have presented (see Ignaczak and Hetnarski [10]). Furukawa et al. used the fundamental equations of generalized thermoelasticity introduced by Noda et al. [11], which include the Lord-Shulman theory and Green-Lindsay theory, and analyzed the one- and two-dimensional problems for plate, for example. In this paper, we treat an isotropic and homogeneous solid sphere. We use the fundamental equations of generalized thermoelasticity which include two theories. The effects of the thermo-mechanical coupling and the relaxation times on the stress focusing phenomena are examined.

2. Analysis

We consider a solid sphere under the suddenly rise of temperature at the free surface at time \( t = 0 \). The fundamental equations of generalized thermoelasticity are follows:

(1) heat conduction equation

\[
\kappa \nabla^2 T - (T - T_0 + t_0 T_{,t})_{,t} = \frac{\delta}{\zeta \alpha} \left[ u_{,r} + \frac{2}{r} u + \delta_{kk} t_0 (u_{,r} + \frac{2}{r} u)_{,t} \right]_{,t} \tag{1}
\]

(2) equation of motion represented by displacement

\[
u_{,rr} + \frac{2}{r} u_{,r} - \frac{2}{r^2} u - \zeta \alpha (T - T_0 + \delta_{kk} t_1 T_{,t})_{,r} = \frac{1}{\nu_e} u_{,tt} \tag{2}
\]

(3) stress-strain-temperature equation

\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\varphi\varphi}
\end{bmatrix} = 2\mu \begin{bmatrix}
u & \bar{u} \\
\bar{u} & r
\end{bmatrix} + \lambda (u_{,r} + \frac{2}{r} u) - (3\lambda + 2\mu)\alpha (T - T_0 + \delta_{kk} t_1 T_{,t}) \tag{3}
\]

\]}
where \( \nu^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \), \( T \) is temperature, \( u \) is displacement of radial direction, \( \sigma_{rr} \) and \( \sigma_{\phi\phi} \) are radial and hoop stress components, respectively, \( \kappa \) is thermal diffusivity, \( \alpha \) is coefficient of linear thermal expansion, \( v_e = \sqrt{(\lambda + 2\mu)/\rho_0} \) is longitudinal wave speed, \( \rho_0 \) is density, \( \lambda \) and \( \mu \) are Lame constant, \( \zeta = (1 + \nu)/(1 - \nu) \), \( \nu \) is Poisson ratio, \( t_0 \) and \( t_1 \) are relaxation times, and \( \delta \) is thermomechanical coupling parameter. The comma denotes the differentiation with following variable. In these equations, \( \delta_{ik} \) is Kronecker delta. When we put \( k=1 \) and \( k=2 \), these equations are coincided to the equations for Lord-Shulman theory and Green-Lindsay theory, respectively.

We introduce the following dimensionless quantities for the convenience.

\[
\begin{align*}
\rho &= \frac{r}{a}, \quad \beta = \frac{K}{v_e a}, \quad \tau = \frac{v_e t}{a}, \\
\tau_0 &= \frac{v_e t_0}{a}, \quad \tau_1 = \frac{v_e t_1}{a}, \quad U = \frac{u}{a \alpha (T_1 - T_0)}
\end{align*}
\]

From Eq. (4) we obtain the equations of dimensionless quantities.

\[
\begin{align*}
\nabla^2 \theta - \frac{1}{\beta} (\theta + \tau_0 \theta, \tau) &= \frac{\delta}{\zeta \beta} \left[ U, \rho + \frac{2}{\rho} U + \delta_{1k} \tau_0 (U, \rho + \frac{2}{\rho} U), \tau \right] \\
\nabla^2 U - \frac{2}{\rho^2} U - U_{\tau \tau} &= \zeta (\theta + \delta_{2k} \tau_1 \theta, \tau), \rho \\
\begin{bmatrix} \sigma_{rr}^{*} \\ \sigma_{\phi\phi}^{*} \end{bmatrix} &= \left[ \begin{array}{c} 1 \\ \eta \end{array} \right] (U, \rho + \frac{2}{\rho} U) + \left[ \begin{array}{c} 2 \\ -1 \end{array} \right] (\eta - 1) \frac{U}{\rho} - \zeta (\theta + \delta_{2k} \tau_1 \theta, \tau)
\end{align*}
\]

where

\[
\eta = \frac{\nu}{1 - \nu}, \quad \nu^2 = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial}{\partial \rho}
\]

Laplace transformed equations under the initial condition

\( \tau = 0: \quad U = U, \theta = 0, \quad \theta = \theta, \tau = 0 \)

are as follows.

\[
\begin{align*}
\nabla^2 \theta^{*} - \frac{(1 + \tau_0 p)}{\beta} p \theta^{*} &= \frac{\delta}{\zeta \beta} (U, \rho + \frac{2}{\rho} U^{*})(1 + \delta_{1k} \tau_0 p) p \\
\nabla^2 U^{*} - \left( \rho^2 + \frac{2}{\rho^2} \right) U^{*} &= \zeta (1 + \delta_{2k} \tau_1 p) \theta^{*}, \rho \\
\begin{bmatrix} \sigma_{rr}^{*} \\ \sigma_{\phi\phi}^{*} \end{bmatrix} &= \left[ \begin{array}{c} 1 \\ \eta \end{array} \right] (U, \rho + \frac{2}{\rho} U^{*}) + \left[ \begin{array}{c} 2 \\ -1 \end{array} \right] (\eta - 1) \frac{U^{*}}{\rho} - \zeta (1 + \delta_{2k} \tau_1 p) \theta^{*}
\end{align*}
\]

where the astrisk denotes the Laplace transform and \( p \) is Laplace parameter.

From Eqs (9) and (10), we obtain the equation for displacement.
\[(\nabla^2 - \frac{2}{\rho^2} - k_1^2)(\nabla^2 - \frac{2}{\rho^2} - k_2^2)\mathbf{U}^* = 0\] (12)

where \(k_1\) and \(k_2\) are positive roots of the following characteristic equation
\[k^4 - p(B_0 + B_1)k^2 + p^3(1 + \tau_0)p/\beta = 0\] (13)

with
\[B_0 = 1 + \left[\tau_0 + \delta(\delta_2 + \delta_3\tau_0)\right]/\beta, \quad B_1 = (1 + \delta)/\beta\]

The solution of the differential equation (12) is
\[U^* = -\zeta(1 + \delta_2k_1\tau_1p)\sum_{i=1}^{2}k_i^2\sinh k_i\rho - k_i\rho \cosh k_i\rho\] (14)

where \(A_1\) and \(A_2\) are unknowns determined from boundary conditions.

Similarly, the temperature and stresses are shown in
\[\theta^* = \sum_{i=1}^{2}(k_i^2 - p^2)\frac{A_i}{\rho^2}\sinh k_i\rho\] (15)

\[\sigma_{rr}^* = \zeta(1 + \delta_2k_1\tau_1p)\sum_{i=1}^{2}A_i[p^2\sinh k_i\rho + \frac{2(1-\eta)}{\rho^2}(\sinh k_i\rho - k_i\rho \cosh k_i\rho)]\] (16)

\[\sigma_{\varphi\varphi}^* = \zeta(1 + \delta_2k_1\tau_1p)\sum_{i=1}^{2}A_i[(k_i^2(\eta-1) + p^2)\sinh k_i\rho - \frac{1-\eta}{\rho^2}(\sinh k_i\rho - k_i\rho \cosh k_i\rho)]\]

The dimensionless boundary conditions of Laplace transformed domain are
\[\rho = 1: \quad \theta^* = \frac{1}{\rho}, \quad \sigma_{rr}^* = 0\] (17)

and unknowns \(A_1\) and \(A_2\) can be determined.

### 3. Numerical examples

![Figure 1 The radial stress distributions.](image)
We calculate for \( \nu = 0.3 \) and \( \beta = 0.1 \) in the context of Lord-Shulman theory. Figure 1 shows the radial stress distribution for \( \delta = 0.02 \) and \( \tau_0 = 0.02 \). From this figure, the peak stresses become large negative values near the center when the time \( \tau < 1 \), and gradually small positive values for \( \tau > 1 \). Figure 2 shows the time deviations of radial stress at the position \( \rho = 0.05 \) closely near the center for \( \delta = 0.02 \) and \( \tau_0 = 0.02 \). The stress accumulates when the stress wave is reached at the position \( \rho = 0.05 \). From further considerations, the order of stress singularity shows the inverse of radius, that is, \( 1/\rho \) for three theories (uncoupled, coupled and genralized cases).

4. References