STRESS PROPAGATION IN SEMI-INFINITE PLANAR HEAP OF GRANULAR MEDIUM POSSESSING SELF-WEIGHT IN LOOSE CONDITION

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1. Summary
Linear and nonlinear stress propagations in the problem of two-dimensional symmetrical sand pile were investigated. The hyperbolic-type differential equations were formulated under the criterion of self-weight loading. This study shows that the admissible stress solution can be obtained in wave-like equation by combining the differential equilibrium equations and the local stress conditions with the boundary conditions. Unlike the linear stress propagation which appears in straight line, the nonlinear stress propagation appears in smooth curves of principal stress directions which are regarded as nests of major and minor arches formed in granular media.

2. Introduction
The local stress conditions in an ideally granular medium make it possible to obtain the admissible stress fields for many problems. The conditions described in Fig.1 provide the basic assumption and geometry conditions of semi-infinite planar sand pile. The local stress conditions in rectangular coordinate system are defined from four closures which are constant stress ratio [1,2], fixed principal axes [3] and oriented stress linearity [4] as well as polarized principal axes [5] which is developed by authors. Unlike polarized principal axes, other previous closures are piecewise bi-linear functions, therefore the planar heap is separated to limit state and below-limit state regions. Stresses of both regions are connected along the separated boundary. However, in the closure of polarized principal axes the limit state is existed along the slope surface only. The results of this study would give rise to define initial stresses of natural slopes in loose condition.

Figure 1 The ideal heap of dry sand in loose condition, forming in a long prismatic shape of semi-infinite planar heap with two symmetrical slopes inclined with angle of repose. The equilibrium of a granular heap with the sides of which are free from traction is considered under planar geometry. Without compaction, granular medium will cascade down once the slope exceeds the angle of repose therefore heap will grow and reach the unique steady state, having the shape unchanged.
3. Background

The stress continuity equations in horizontal and vertical directions can be given in form of implicit function $E_x$ and $E_z$ where $\sigma_{xx}$ and $\sigma_{zz}$ are normal stresses in horizontal and vertical direction, $\sigma_{xz}=\sigma_{zx}$ is shear stress and $\gamma$ is a unit weight.

$$E_x(x,z) = \partial_x \sigma_{xx} + \partial_z \sigma_{xz} = 0$$  \hspace{1cm} (1)

$$E_z(x,z) = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \gamma = 0$$  \hspace{1cm} (2)

The second derivatives of Eq.(1) and Eq.(2) with respect to $x$ and $z$ are shown in Eqs.(3)-(4) and Eqs.(5)-(6) respectively.

$$\partial_x E_x(x,z) = \partial^2_{xx} \sigma_{xx} + \partial^2_{xz} \sigma_{xz} = 0$$  \hspace{1cm} (3)

$$\partial_x E_z(x,z) = \partial^2_{xx} \sigma_{xz} + \partial^2_{xz} \sigma_{zz} = 0$$  \hspace{1cm} (4)

$$\partial_z E_x(x,z) = \partial^2_{xz} \sigma_{xx} + \partial^2_{zz} \sigma_{zz} = 0$$  \hspace{1cm} (5)

$$\partial_z E_z(x,z) = \partial^2_{xz} \sigma_{xz} + \partial^2_{zz} \sigma_{zz} = 0$$  \hspace{1cm} (6)

Let us formulate a linear combination of differential equations from Eq.(1)-(6) to an implicit function $\Xi = \Xi(x,z)$ using arbitrary coefficients $a_1, a_2, a_3, a_4, a_5, a_6$. The value of $\Xi$ is zero due to a summation of the differential equations which are regarded as implicit functions.

$$\Xi(x,z) = a_1 E_x + a_2 E_z + a_3 \partial_x E_x + a_4 \partial_x E_z + a_5 \partial_z E_x + a_6 \partial_z E_z$$  \hspace{1cm} (7)

Supposing that $\Xi$ is a continuous function, the order of $\partial_k \sigma_{ij}=\partial_k \sigma_{ij}=\partial_k \sigma_{ji}$ are satisfied. Hence, Eq.(7) can be decomposed to Eq.(8) using the following operators.

$$\Xi(x,z) = \left( a_1 \partial_x + a_2 \partial_z \right) \sigma_{xx} + \left( a_3 \partial_x + a_4 \partial_z \right) \sigma_{xz} + \left( a_5 \partial_x + a_6 \partial_z \right) \sigma_{zz}$$

$$+ \left( a_1 \partial_x + a_2 \partial_z \right) \sigma_{z} - a_3 \gamma$$  \hspace{1cm} (8)

For an arbitrary $\sigma_{xz}$, $\Xi$ can be freed from $\sigma_{xz}$ by selecting $a_1=a_2=a_4=0$ and $a_3=-a_6$. Providing that $a_6$ is an arbitrary coefficient, the following partial differential equation can be proven.

$$\partial^2_{zz} \sigma_{zz} - \partial^2_{xx} \sigma_{xx} = 0$$  \hspace{1cm} (9)

The above partial differential equation can be solved if a specific stress condition linking all stress components is given. According to the earlier researches listed in Table 1, three stress conditions are given under closures of constant stress ratio [1,2], fixed principal axes [3] and oriented stress linearity [4]. Correspondingly, with manipulation of Eqs.(1)-(2), Eq.(9) can formulate a typical form of a second order partial differential equation for a general stress $\sigma_{ij}$, representing any terms of $\sigma_{xx}$, $\sigma_{xz}$ and $\sigma_{zz}$.

$$\left( \partial_x + c_1 \partial_x \right) \left( \partial_z + c_2 \partial_z \right) \sigma_{ij} = \left( \partial^2_{xx} + (c_1 + c_2) \partial^2_{xz} \right) \sigma_{ij} = 0$$  \hspace{1cm} (10)

where $c_1$ and $c_2$ denote positive and negative coefficients respectively. By equation manipulation, $c_1$ and $c_2$ can be obtained. Eq.(10) can be classified as a hyperbolic type differential equation because $(c_1+c_2^2)-4c_1c_2>0$. Despite of $c_1=c_2$, Bouchaud et al. [2] observed the form expressed in Eq.(10) looks like a wave equation in two dimensions for a physical space $x$ and $z$. For $c =c_1=c_2$, ...
it becomes an ordinary wave equation. The wave equation is a typical example of a hyperbolic partial differential equation with a parameter $c$ equal to propagation speed of the wave. According to Bouchaud et al. [2], the characteristics of the wave-like Eq.(10) can be represented by two straight lines of slope $c_1$ and $c_2$ which coincide with the principal axes of stresses. Therefore, self-weight transfer is propagated along major and minor arches representing major and minor axes of principal stresses which are viewed as load-bearing structures composed of a set of nested arches.

Table 1 Various stress conditions in semi-infinite planar heap of granular medium.

<table>
<thead>
<tr>
<th>Stress conditions</th>
<th>Partial differential equations</th>
<th>Coefficients of wave-like equation</th>
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<tbody>
<tr>
<td>Constant stress ratio $\sigma_{xx} = K \sigma_{zz}$</td>
<td>$(\partial_{zz}^2 - K \partial_{xx}^2) \sigma_{y} = 0$</td>
<td>$c_1 = \sqrt{K}, c_2 = -\sqrt{K}$</td>
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<tr>
<td>Fixed principal axes $\sigma_{xx} = \sigma_{zz} - 2 \sigma_{xz} \tan \phi$</td>
<td>$(\partial_{zz}^2 - 2 \tan \phi \partial_{xz}^2 - \partial_{xx}^2) \sigma_{y} = 0$</td>
<td>$c_1 = \frac{1 - \sin \phi}{\cos \phi}, c_2 = \frac{1 + \sin \phi}{\cos \phi}$</td>
</tr>
<tr>
<td>Oriented stress linearity $\sigma_{xx} = \eta \sigma_{zz} + \mu \sigma_{xz}$</td>
<td>$(\partial_{zz}^2 + \mu \partial_{xz}^2 - \eta \partial_{xx}^2) \sigma_{y} = 0$</td>
<td>$c_1 = \frac{\mu + \sqrt{\mu^2 + 4 \eta}}{2}, c_2 = \frac{\mu - \sqrt{\mu^2 + 4 \eta}}{2}$</td>
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4. Methods

In order to describe nonlinearity of principal stress orientation, the new closure of polarized principal axes is employed (Pipatpongsa et al. [5]) in which stress condition in rectangular coordinate is given as follows.

$$\sigma_{xx} = \sigma_{zz} - 2 \frac{z}{x} \sigma_{xz}$$

(11)

One might observe that Eq.(11) can be reduced to the closure of fixed principal axes when $z/x = \tan \phi$ is thoroughly kept constant. Eq.(7) substituted by Eq.(11) can remove the components of $\sigma_{xz}$. By enforcing $a_1 = a_2 = a_3 = a_6 = 0$ and $a_5 = a_4$, the further manipulation can be arranged to remove the components of $\sigma_{zz}$, therefore the partial differential for $\sigma_{zz}$ is obtained.

$$-\partial_{,x} E_z + \partial_{,z} E_x = (\partial_{zz}^2 - 2 \partial_{xz}^2 \frac{z}{x} - \partial_{xx}^2) \sigma_{zz} = 0$$

(12)

Unlike the linearity principal stress orientation, the partial differential for nonlinearity can be formulated only for $\sigma_{zz}$. Therefore, once the solution for $\sigma_{zz}$ is found, the solutions for $\sigma_{xx}$ and $\sigma_{zz}$ can be derived from Eqs.(1)-(2) using $\sigma_{zz}$. Further expansion of Eq.(12) can be formulated.

$$\left( \frac{\partial_{xx}^2}{x} - 2 \frac{z}{x} \partial_{xz}^2 - \partial_{xz}^2 \right) \left( \frac{x^2 + z^2}{x} - \sigma_{xx} \right) = \left( \partial_{,z} + c_1(x,z) \partial_{,x} \right) \left( \partial_{,z} + c_2(x,z) \partial_{,x} \right) \left( \frac{x^2 + z^2}{x} \sigma_{xx} \right)$$

(13)

where $c_1(x,z) = -z/x + \sqrt{1 + (z/x)^2}$ and $c_2(x,z) = -z/x - \sqrt{1 + (z/x)^2}$.
5. Results

The resulting admissible stress fields can be solved from the wave-like equation with stress-free boundary at the slope surface, using $c_1$ and $c_2$ of which are equivalent to propagation velocities. As with the linearity models, the solutions are separated to inner elastic and outer plastic regions with merge at a certain slope, therefore piecewise linear stress profiles are obtained. Stress solutions under the closures of oriented stress linearity provided by Wittmer et al. [4] covers those of constant stress ratio and fixed principal axes whose directions of principal axes are plotted in Figs.2-3 using $\phi=33^\circ$. Principal directions shown in Fig.3 are illustrated for the solution obtained under the closure of polarized principal axes. It can be observed in Fig.3 that self-weight is propagated along nonlinear load-bearing structures with hump profile over the width of granular heaps. Moreover, plastic region is saturated only along the slope surface, therefore, smooth transition of arches transferring own weight can be seen.

6. Conclusions

This study shows that stress field solutions under nature law of friction can be derived through wave-like equations. Linear and nonlinear stress propagations due self-weight transfer in cohesionless granular media were investigated. Nests of principal stresses were illustrated in comparison among various stress conditions. It was found that the linear stress propagation follows straight arches with a certain elastic-plastic slope inside, while the nonlinear stress propagation follows smooth curves of arches with elastic-plastic slope along the slope surface.

7. References